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FORCED RESPONSE ANALYSIS OF HIGH-MODE VIBRATIONS FOR MISTUNED BLADED DISCS WITH EFFECTIVE REDUCED-ORDER MODELS

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ABSTRACT

This paper is focused on the analysis of effects of mistuning on the forced response of gas-turbine engine bladed discs vibrating in the frequency ranges corresponding to higher modes. For high modes considered here the blade aerofoils are deformed during vibrations and the blade mode shapes differ significantly from beam mode shapes. A model reduction technique is developed for the computationally efficient and accurate analysis of forced response for bladed discs vibrating in high frequency ranges. The high-fidelity finite element model of a tuned bladed disc sector is used to provide primary information about dynamic properties of a bladed disc and the blade mistuning is modelled by specially defined mistuning matrices. The forced response displacement and stress amplitude levels are studied. The effects of different types of mistuning are examined and the existence of high amplifications of mistuned forced response levels is shown for high-mode vibrations: in some cases, the resonance peak response of a tuned structure can be lower than out-of-resonance amplitudes of its mistuned counterpart.

INTRODUCTION

Bladed discs of gas-turbine engines and turbomachines for power are usually designed to have identical blade geometry, material properties and contact characteristics of joints. In reality, however, there are inevitable and, usually, small differences in blade characteristics which occur due to blade geometry scatters within manufacture tolerances, assembling blades in a disc, blade wear during engine service, and other causes. Such differences in blades result in the scatter of natural frequencies and mode shapes of individual blades which is customarily called 'mistuning'. Mistuning, even very small, can drastically change dynamic properties of a bladed disc, which usually leads to occurrence of a large number of resonance peaks, scatter of resonance peak amplitudes for different blades of a bladed disc and, the most important, to a large increase of the forced response levels (Refs. [1]-[5]). The maximum amplitudes of a mistuned bladed disc can be several times larger than that of the tuned one and this can lead to high cycle fatigue (HCF) failures of the blades, if the design process relies on the tuned bladed disc analysis (see Refs.[6] and [7]). High vibration stress levels of a mistuned bladed disc are highly sensitive to the blade mistuning patterns. In order to improve the reliability of bladed discs, it is important to model and analyse accurately mistuning effects at the design stage.

In the past, various models, including the lumped mass models, beam and plate models, and realistic finite element (FE) models, were developed and examined for analysis of forced response of mistuned bladed discs. The lumped parameter models generally consider each blade and a corresponding disc sector as lumped masses connected by springs. These models were extensively used to explore fundamental, qualitative properties of the mistuned forced response in the early studies, since relatively small calculation efforts were required ([8] and [9]). In attempts to increase the modelling accuracy, beam blade and disc plate models were applied ([10]-[11]). These models played an important role in mistuning research from the 1970s to the 1990s, as it could simulate some vibration modes of real blades, such as bending and torsion. Both models mentioned above, although highly simplified, were useful for qualitative analysis leading to understanding general trends in the mistuning effects on bladed disc vibrations. Yet, they cannot be used for assessment of mistuning effects of mistuning and accurate quantitative prediction of the vibration response for an actual bladed disc in a gas-turbine engine.

The high-fidelity FE models started to be used in the calculation of vibration characteristics of tuned and then mistuned bladed discs with the development of computing resources. A sector model of a tuned bladed disc, comprising a blade and an adjoining disc sector, can accurately represent a whole bladed disc. In industrial applications, such a sector model usually contains from hundred thousand to millions degrees of freedoms (DOFs). Blade mistuning destroys the cyclic symmetry of a bladed disc and, therefore, a full model, including all sectors, is required for the mistuned bladed disc analysis. Therefore, the total number of DOFs in a mistuned structure to be analysed is drastically increased and this number is usually too large to allow the calculation of mistuned bladed disc directly and several reduction analyses methods have been developed recently.

Methods based on the component mode synthesis (CMS) method were developed in Refs. [12]-[14]. CMS is applied individually for each blade and disc sector in order to reduce the number of DOFs for each component of a bladed disc.

Another approach used for the model reduction for mistuned bladed discs is based on using modal properties of a whole structure. An example of using this approach for the model reduction was developed in Ref. [15]. A selected set of tuned system modes is used here as “nominal modes” to provide a basis for representing the vibration of mistuned bladed discs and the method is called here a subset of nominal modes (SNM) method. Making further simplification and using only one family of modes, Feiner and Griffin (Ref. [16]) subsequently derived a so-called “fundamental model of mistuning” (FMM) applicable for a case of an isolated family of blade-dominated modes. A comparison of properties of two mentioned above approaches for the model reduction is made in Ref. [17].

In addition, a mistuning analysis method based on an exact relationship between the response levels of the tuned and mistuned bladed discs was proposed in Ref. [18]. The high efficiency of the method was achieved using the possibility to calculate the forced response for only a small subset of DOFs: those where the forced response levels are analysed and those where the mistuning modification are applied.

The reduction techniques for analysis of geometric mistuning, where the blade geometry mistuning is modelled by blade geometry variation explicitly, are developed in Refs. [19] and [20].

Most numerical studies of forced response of mistuned bladed discs were focused on the analysis of vibration in relatively low frequency ranges which correspond to several first blade modes. Typically, the analysed frequency range included first or second bending or torsional modes.

There is no high-fidelity analysis performed for a mistuned bladed disc when the blades vibrate in higher modes where the mode shapes are complex and blade aerofoils are deforming. It is well known that the excitation forces in gas-turbine engines can have a very dense frequency spectrum (Ref. [21]). Similarly, the spectrum of the natural frequencies of typical bladed discs is also very dense. This affects more significantly high modes, where even 1% mistuning can cause large resonance frequency variations and the scatter of blade frequencies in high modes can be more than 100Hz. The large scatter of blade frequencies can lead to a wide frequency range where all resonance peaks of a family of modes of interest are located and, moreover, resonance peaks of different families of modes become close and can interact. The coupling of vibrations of mistuned blades through disc is dependent on the blade mode analysed. It is expected that such coupling should differ significantly for vibration in high mode frequency ranges from those of well-studied cases of low-mode vibrations. Furthermore, since high vibration amplitudes for blades vibrating in higher modes can be localised in a very restricted blade area, it is necessary to pay a special attention to blade mistuning modelling and, accordingly, to a choice of the appropriate model reduction technique.

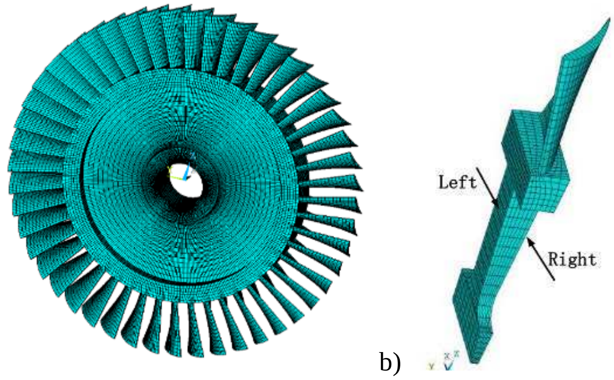
In this paper, an unexplored problem of analysis for bladed discs excited in high-mode frequency ranges is considered. A model reduction technique is developed for computationally efficient analysis of forced response for high-frequency vibration of mistuned bladed discs using high-fidelity models. The technique is based on the known idea of using modal properties of a tuned structure for the condensation of a mistuned structure, but modifications are introduced to increase the efficiency of calculations. The mistuning effects on high-mode forced responses and stresses of bladed discs are discussed in detail. A method to decompose the mistuned response and to evaluate contributions of tuned modes in the mistuned forced response is also presented, which helps to establish and control the dominant mode in the mistuned response. Moreover, an efficient method to calculate the stresses and strains is proposed, and relationship between amplitude and stress is also examined. In the end, the effects of mistuning element distribution on response and statistical properties of the mistuned forced response are analysed for different engine orders.

METHOD FOR ANALYSIS OF FORCED RESPONSE OF MISTUNED BLADED DISCS

The finite element model of a tuned bladed disc sector is used to provide primary information about modal properties of a tuned bladed disc, which are used in the analysis of mistuning structures. The blade mistuning is then modelled by specially defined mistuning matrices simulating experimentally measured scatter of dynamic properties of blades.

A tuned bladed disc possesses the cyclic symmetry property: its geometry and other properties are unchanged by rotations about the cyclic symmetry axis by a certain angle.

Modal properties of a tuned structure



a) b)

Fig. 1 A tuned bladed disc: a) a whole bladed disc; b) a sector

Such a bladed disc can be modelled and calculated using just a sector finite element model, which must ensure the two boundary areas separating the sector from neighbouring in the whole structure have identical mesh, as shown in Fig. 1. A numerical analysis of modal characteristics of a tuned bladed disc can be performed using a sector model only, which represents exactly a whole bladed disc (see Ref.[22]). The analysis is performed with special cyclic symmetry conditions applied on displacements and forces acting at the boundaries at which one sector interacts with the neighbouring sectors. Such cyclic symmetry boundary conditions have the form:

- for displacements:

$$\boldsymbol{X}_{right} = e^{i\alpha k}\boldsymbol{X}_{left} \tag{1}$$

- for forces:

$$\boldsymbol{Q}_{right} = e^{i\alpha k}\boldsymbol{Q}_{left} \tag{2}$$

where $\boldsymbol{X}$ and $\boldsymbol{Q}$ are vectors of displacement and internal forces and subscripts 'right' and 'left' indicate DOFs lying on the right and left boundaries separating the individual sector from the rest of the bladed disc, $i=\sqrt{-1}$, $\alpha = 2\pi / N_B$ is the sector angle, N_B is the total number of sectors in the bladed disc, and k is the number of waves of deformation along bladed disc circumference. Using these equations, DOFs at one of the boundaries can be excluded and, as a result, we can obtain for a modal analysis of the bladed disc the following equation for a sector:

$$\boldsymbol{K}_k^S \boldsymbol{\Phi}_k^S = \boldsymbol{M}_k^S \boldsymbol{\Phi}_k^S \boldsymbol{\Lambda}_k^S \tag{3}$$

where the stiffness, $\boldsymbol{K}_k^S\left(N_S \times N_S\right)$, and mass, $\boldsymbol{M}_k^S\left(N_S \times N_S\right)$, matrices of a sector are Hermitian matrices depending on the wave number k. $\boldsymbol{\Lambda}_k^S\left(n_S \times n_S\right)$ and $\boldsymbol{\Phi}_k^S\left(N_S \times n_S\right)$ are matrices of eigenvalues and mode shapes containing first n_S natural frequencies and mode shapes for each k value and N_S is the number of DOFs in a sector. The natural frequencies corresponding to $k \neq 0$ or $N_B/2$ are double: the pair of different mode shapes correspond to each such double frequency. These pairing mode shapes are complex conjugates. Eq.(3) is solved for every value of wave number k from 0 to $N_B/2$ in order to obtain natural frequencies and mode shapes a tuned bladed disc for all possible values. The mode shapes of a whole bladed disc are obtained using the cyclic symmetry properties of the mode shapes for each wave number, k, in the form:

$$\underset{(N\times n_S)}{\boldsymbol{\Phi}_k^T} = \mu_k \left[\boldsymbol{\Phi}_k^S, \mathrm{e}^{i\alpha k}\,\boldsymbol{\Phi}_k^S, \mathrm{e}^{i2\alpha k}\,\boldsymbol{\Phi}_k^S ..., \mathrm{e}^{i\alpha k(N_B-1)}\,\boldsymbol{\Phi}_k^S\right]^T \tag{4}$$

where $N = N_S N_B$ is the total number of DOFs in the whole bladed disc; $\mu_k = 1\big/\sqrt{N_B}$; for $k=0$; $\mu_k = -1\big/\sqrt{N_B}$ for $k = N_B/2$ and $\mu_k = \sqrt{2/N_B}$ for all other values of k. The coefficient, μ_k, is introduced here to keep the mass normalisation properties for the mode shapes of a whole structure. The mode shapes of a whole bladed disc, $\boldsymbol{\Phi}\left(N\times n\right)$ (where $n = n_S N_B$ is the total number of modes used in the analysis) are combined from mode shapes obtained for each wave number, $\boldsymbol{\Phi}_k\left(N \times n_S\right)$ in the following form:

- for a case when the number of sectors is odd:

$$\boldsymbol{\Phi} = \left[\boldsymbol{\Phi}_0 \;\vdots\; \mathrm{Re}(\boldsymbol{\Phi}_1) \;\vdots\; \mathrm{Im}(\boldsymbol{\Phi}_1) \;\vdots\; \cdots \;\vdots\; \mathrm{Re}(\boldsymbol{\Phi}_{N_B/2}) \;\vdots\; \mathrm{Im}(\boldsymbol{\Phi}_{N_B/2})\right] \tag{5}$$

- for a case when the number of sectors is even:

$$\boldsymbol{\Phi} = \left[\boldsymbol{\Phi}_0 \;\vdots\; \mathrm{Re}(\boldsymbol{\Phi}_1) \;\vdots\; \mathrm{Im}(\boldsymbol{\Phi}_1) \;\vdots\; \cdots \;\vdots\; \mathrm{Im}(\boldsymbol{\Phi}_{N_B/2-1}) \;\vdots\; \boldsymbol{\Phi}_{N_B/2}\right] \tag{6}$$

It should be noted that the mode shapes of the whole bladed disc, represented by vector-columns of matrix $\boldsymbol{\Phi}$, are obtained here in coordinate systems which are individual for each sector, namely, the local coordinate system for *j*-th sector is rotated around the cyclic symmetry axis by angle $\alpha\left(j-1\right)$ with respect to first sector.

Calculation of modal properties of a mistuned structure

The eigenproblem for a mistuned structure has the form:

$$\left[\boldsymbol{K}+\delta\boldsymbol{K}\right]\tilde{\boldsymbol{\Phi}}=\left[\boldsymbol{M}+\delta\boldsymbol{M}\right]\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Lambda}} \tag{7}$$

where mistuning is introduced by modifications of the mass matrix, $\delta\boldsymbol{M}$, and the stiffness matrix, $\delta\boldsymbol{K}$. Direct solution of the eigenproblem for a whole bladed disc is usually too time consuming due to very large sizes of the matrices involved, so mistuned mode shapes, $\tilde{\boldsymbol{\Phi}}$, are expressed, similar to Refs.[15] and [17], as a linear combination of tuned bladed disc mode shapes:

$$\tilde{\boldsymbol{\Phi}}=\boldsymbol{\Phi}\boldsymbol{c} \tag{8}$$

where $\boldsymbol{c}$ is the matrix of coefficients of mistuned mode shape expansion over tuned mode shapes. Then, substituting Eq. (8) in Eq.(7), the following eigenproblem can be formulated:

$$\left[\boldsymbol{\Lambda}+\boldsymbol{\Phi}^T\delta\boldsymbol{K}\boldsymbol{\Phi}+\right]\boldsymbol{c}=\left[\boldsymbol{I}\quad\boldsymbol{\Phi}^T\delta\boldsymbol{M}\boldsymbol{\Phi}\right]\boldsymbol{c}\tilde{\boldsymbol{\Lambda}} \tag{9}$$

The solution of this equation gives us eigenvalues of a mistuned structure, $\tilde{\boldsymbol{\Lambda}}\left(n\times n\right)$, and matrix of the coefficients of the expansion of mistuned mode shapes over mode shapes of a tuned structure, $\boldsymbol{c}\left(n\times n\right)$. This matrix can be calculated to satisfy the normalization condition:

$$\boldsymbol{c}^T\left[\boldsymbol{I}+\boldsymbol{\Phi}^T\delta\boldsymbol{M}\boldsymbol{\Phi}\right]\boldsymbol{c}=\boldsymbol{I} \tag{10}$$

Then the mistuned mode shapes obtained from Eq.(8) are also mass-normalized:

$$\tilde{\boldsymbol{\Phi}}^T\left(\boldsymbol{M}+\delta\boldsymbol{M}\right)\tilde{\boldsymbol{\Phi}}=\boldsymbol{c}^T\left[\boldsymbol{I}+\boldsymbol{\Phi}^T\delta\boldsymbol{M}\boldsymbol{\Phi}\right]\boldsymbol{c}=\boldsymbol{I} \tag{11}$$

Forced response analysis of a mistuned structure

The equation of motion for a mistuned structure has the form:

$$\tilde{\boldsymbol{K}}x+\tilde{\boldsymbol{D}}\dot{x}+\tilde{\boldsymbol{M}}\ddot{x}=\boldsymbol{f}(t) \tag{12}$$

For a case of harmonic excitation $\boldsymbol{f}(t)=\boldsymbol{F}e^{i\omega t}$ then the forced response will be also harmonic $\boldsymbol{x}(t)=\boldsymbol{X}e^{i\omega t}$ and the forced response amplitudes are obtained from:

$$\left[\tilde{\boldsymbol{K}}+i\tilde{\boldsymbol{D}}-\omega^2\tilde{\boldsymbol{M}}\right]\boldsymbol{X}=\boldsymbol{F} \tag{13}$$

For the engine order excitation by k -th excitation harmonic, the phase shift between neighbouring sectors of a bladed disc is equal to $e^{i\alpha k}$ and the force vector for the whole structure takes the following form:

$$\boldsymbol{F}=\left\{\boldsymbol{F}^S,e^{i\alpha k}\boldsymbol{F}^S,\ldots,e^{i\alpha k(N_B-1)}\boldsymbol{F}^S\right\}^T \tag{14}$$

Using the well-known expression for FRF matrix, the amplitudes can be calculated through modal characteristics of the mistuned bladed disc as:

$$\boldsymbol{X}=\sum_{j=1}^{n}\frac{\tilde{\boldsymbol{\phi}}_j^T\boldsymbol{F}}{(1+i\tilde{\eta}_j)\tilde{\lambda}_j-\omega^2}\tilde{\boldsymbol{\phi}}_j \tag{15}$$

where $\tilde{\eta}_j$ is the modal damping factor for j -th mode, $\tilde{\lambda}_j$ is the eigenvalue and $\tilde{\boldsymbol{\phi}}_j$ is the corresponding mode shape of a mistuned structure. The modal damping factors allow for all types of major damping effects: aerodynamic, damping in friction joints and material. An effective method of calculation of the contribution of aerodynamic damping is proposed in Ref.[25]. It should be noted that the blade amplitudes can be calculated using this formula for any set of nodes selected from the whole structure, which reduces significantly the computational time while allowing amplitude evaluation for all nodes of interest.

Tuned mode shapes contributions to forced response of mistuned structure

Mistuned forced response can be expressed as a linear combination of the tuned mode shapes as shown in Eq.(8). In many cases the coefficients of such an expansion give useful information for analysis of the mistuned response. Such coefficients can be efficiently calculated simultaneously with the forced response analysis for each excitation frequency. Substitution of Eq.(8) in Eq.(15) gives:

$$\boldsymbol{X}=\boldsymbol{\Phi}\boldsymbol{c}diag\left(\left[\left(1+i\eta_j\right)\lambda_j-\omega^2\right]^{-1}\right)\tilde{\boldsymbol{F}} \tag{16}$$

where $\tilde{\boldsymbol{F}} = \tilde{\boldsymbol{\Phi}}^T \boldsymbol{F}$ is a vector of modal forces obtained for the mistuned bladed disc and $\boldsymbol{c}$ was obtained earlier from Eq.(9). Since the forced response of mistuned bladed disc can be expressed through tuned mode shapes in the form:

$$\boldsymbol{X} = \boldsymbol{\Phi}\tilde{\boldsymbol{c}} \tag{17}$$

then comparing Eqs.(16) and (17) we can obtain the expression for the vector of contributions of each tuned mode shape in the mistuned response for each excitation frequency, ω:

$$\tilde{\boldsymbol{c}} = \boldsymbol{c}\begin{Bmatrix} \tilde{f}_1 / \left((1+i\eta_1)\lambda_1 - \omega^2 \right) \\ \cdots \\ \tilde{f}_n / \left((1+i\eta_n)\lambda_n - \omega^2 \right) \end{Bmatrix} \tag{18}$$

where $\tilde{f}_j$ ($j = 1..n$) are the components of vector of modal forces, $\tilde{\boldsymbol{F}}$.

Calculation of stresses and strains for a mistuned structure

The equation of motion of a mistuned structure can be derived as:

$$\left[(\boldsymbol{K} + \delta\boldsymbol{K}) + i(\boldsymbol{D} + \delta\boldsymbol{D}) - \omega^2 (\boldsymbol{M} + \delta\boldsymbol{M}) \right] \boldsymbol{X} = \boldsymbol{F} \tag{19}$$

The amplitudes of a mistuned structure, $\boldsymbol{X}$, can be easily obtained from Eq.(15). Then, Eq.(19) can be transformed, by moving the mistuned part to the right hand side:

$$\left[\boldsymbol{K} + i\boldsymbol{D} - \omega^2 \boldsymbol{M} \right] \boldsymbol{X} = \boldsymbol{F} - \left(\delta\boldsymbol{K} + i\delta\boldsymbol{D} - \omega^2 \delta\boldsymbol{M} \right) \boldsymbol{X} = \boldsymbol{F}_{mist} \tag{20}$$

The vector $\boldsymbol{F}_{mist}$ includes the excitation forces and stiffness damping and inertia forces occurring due to mistuning. This vector is calculated using the found displacements of a mistuned bladed disc, $\boldsymbol{X}$. On the left hand side of Eq.(20) all matrices are matrices of a tuned structure. So the analysis of a mistuned structure can be performed for a tuned structure which is subjected to this vector, $\boldsymbol{F}_{mist}$, of non-cyclic loads. This fact allows take advantages of cyclic symmetry properties and perform calculation of stresses, strains and displacements using a sector finite element model of a bladed disc. To do this, the spatial distribution of the non-cyclic loads are expanded in discrete Fourier series over the circumferential coordinate. For a tuned cyclically symmetric bladed disc, the total number of different harmonic in such an expansion is equal to the number of blades. The forced response is then calculated for each harmonic of this expansion and then the responses excited by each harmonic are summed to obtain the forced response for a mistuned bladed disc. The capability to calculate a cyclically symmetric structure under non-cyclic loading is available in commercial FE software packages.

Fast analysis of the resonance peak responses

The knowledge of the mistuned modal properties can be fully used in order to minimise the computation time, which of particular importance in the analysis of mistuned forced response.

Firstly, the mistuned natural frequencies are close to resonance response frequencies and, therefore, the knowledge of their values allows the selection of a set of narrow frequency ranges which include all resonance peaks.

Moreover, comparison of the resonance peak amplitude at the mistuned natural frequencies allows selection of only small subset of resonance peaks which provide higher response levels. As a result of such selection, the number of frequency steps required for the accurate determination the maximum resonance peak can be reduced by orders of magnitude.

Furthermore, the knowledge of distributions of amplitudes for mistuned mode shapes, which are close to the maximum resonance peak, allows the calculation of displacements and stresses only for blades and finite element nodes where the high vibration amplitudes are located. Owing to this, the computation effort required for calculation is significantly reduced.

A typical example of the forced response calculated by a conventional approach using frequency steps which are chosen to be small enough to capture accurately all resonance peaks of a bladed disc comprising 48 blades is shown in Fig. 2. This calculation required approximately 3000 frequency steps and this forced response curve is plotted by black curve. This forced response is compared with the forced response calculated only at the natural frequencies of the mistuned bladed disc. The red dots show amplitudes calculated at frequencies corresponding to all 48 natural frequencies of the family of modes located in this frequency range.

One can see that the amplitudes calculated at the mistuned natural frequencies give good approximations for the resonance peak amplitudes. It is evident that the frequency range around 10 kHz is the most dangerous frequency range and a narrow frequency range can be chosen around this frequency to calculate efficiently the maximum mistuned response: with very small number of frequency steps.

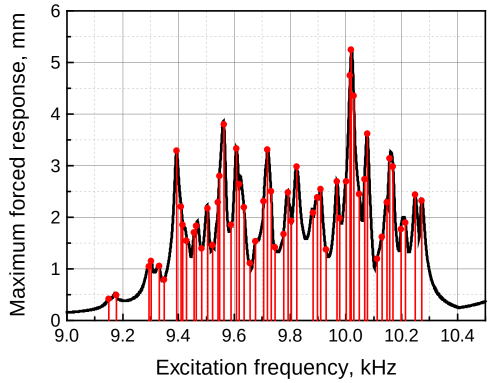


Fig. 2 Selection of narrow frequency ranges for resonance response calculation

Moreover, the blade maximum amplitudes calculated with the method described above are compared with the amplitude distributions determined from 3 mode shapes. The mode shapes are chosen which are the closest to the maximum resonance peak in the frequency range analysed. The comparison of these amplitudes is shown in Fig. 3, where we can notice that the chosen three mode shapes can help to localise the blades with large amplitudes. Owing to this, the mode shapes of the mistuned bladed disc can help in the choice of a set of blades experienced high forced response levels (in the considered case it is the blade number 24) and, therefore, the computation efforts can be reduced significantly by calculation of amplitudes only for these blades.

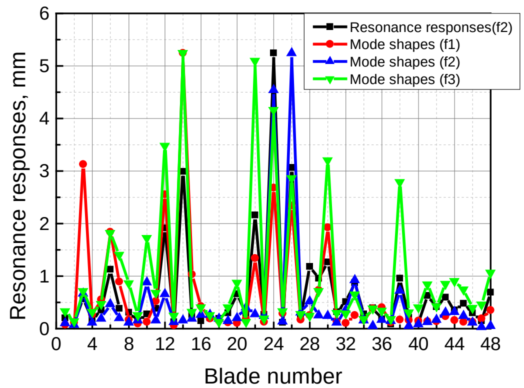


Fig. 3 Maximum resonance amplitudes for all blades

BLADED DISC AND MISTUNING MODELLING

A compressor bladed disc, comprising 48 sectors, was considered. The three-dimensional finite element model used in the analysis is shown in Fig. 1. The whole model comprises about 1,500,000 degrees of freedom and a sector model comprises about 35,000 DOFs. The natural frequencies and mode shapes of this tuned bladed disc were obtained from a sector model for all possible values of nodal diameters (NDs) in this disc, i.e. from 0 to 24. The dependencies of natural frequencies on the number of NDs are plotted in the Fig. 4, where the first 12 natural frequencies are shown for each ND value. It should be noted that, for the analysis of the mistuned forced response, first 32 natural frequencies and mode shapes were calculated for each possible ND number.

Results corresponding to excitation by 6EO and 15EO are mostly presented in this paper: 6EO corresponds to the case when the bladed interaction through disc is significant and 15EO corresponds to the case when disc become stiff and blade vibration coupling is low. These EOs correspond to the 6 and 15 NDs and they are shown by vertical lines in Fig. 4. The excitation forces were applied to all the nodes on the blade surfaces and the forces act in axial and tangential directions.

The analysed frequency ranges include natural frequencies of 6^{th} and higher families of modes. The examples of mode shapes for 6^{th}, 7^{th} and 8^{th} modes of a lone blade are shown in Fig. 5. A structural damping modal factor was assumed to be 0.2% for all modes involved in the forced response analysis.

Due to manufacture scatters and blade wear during service the blades have always scatters in their dynamic properties: so called “mistuning”. The variation of the blade dynamic characteristics is modelled here by adding or subtracting some small lumped masses which are distributed over nodes of finite element models of blades. This approach for the mistuning modelling allows description accurately the scatter of a natural frequency of interest by choosing appropriate values for the added masses to fit the experimentally measured values, when they are available. The effects of different distribution of the mistuning elements over blades on the modal characteristics of mistuned blades and on the mistuned forced response were explored (Fig.6).

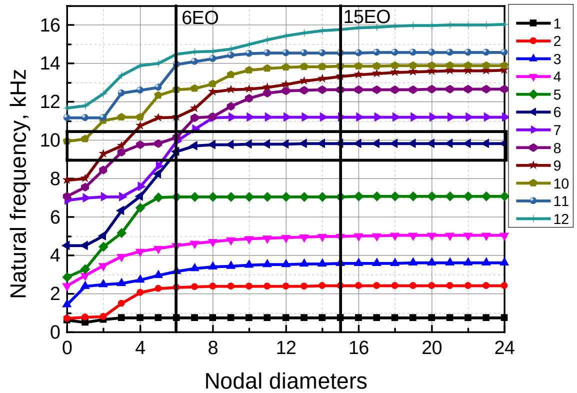


Fig. 4 Natural frequencies of the tuned bladed disc

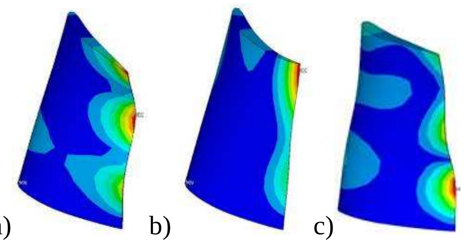

a) b) c)

Fig. 5. Mode shapes of alone blade: a) 6th mode; b) 7th mode; c) 8th mode

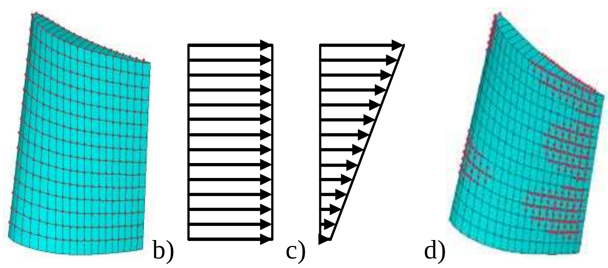

a) b) c) d)

Fig. 6. Mistuning element distributions: a) applied to all nodes; b) uniform distribution along blade length; c) linear distribution along blade length; d) based on mode shapes.

Three different types of mistuning element distributions are considered: (i) mistuning elements are uniformly distributed over nodes of convex and concave blade surfaces (see Fig. 6a,b); (ii) mistuning elements are distributed over linearly along blade length (Fig. 6c) and (iii) mistuned elements are distributed over regions where the mode shape amplitudes are maximum (Fig. 6d). In the first two cases the element mass values were varied along aerofoil width proportionally to the aerofoil thickness - to avoid application of too large masses at thin aerofoil parts.

The mistuning analysis is performed for randomly created blade mistuning patterns keeping similar mass distributions over all individual blades but scaling their mass values to obtain desired blade frequency mistuning. The 6th blade mode was selected to control the blade mistuning, and blade frequency mistuning was generated in the range [-5%, +5%] of the tuned frequency value. The applied mistuning elements affect not only a frequency of the controlled mode, but all blade natural frequencies. The effects of mistuning element distributions on the natural frequencies of all blade modes included in the analysis have been examined.

The deviations of mistuned blade natural frequencies, obtained for three types of mistuning elements discussed above, from values obtained for a tuned blade are plotted in Fig. 7 for first 12 natural frequencies and for two cases: (i) -5% and (ii) +5% mistuning. One can see that first two types of mistuning element distributions provide similar effects on all natural frequencies, while the mode-shape based distributions change differently frequencies of different modes: e.g. 1% shift for 1st mode and 5% for 6th mode.

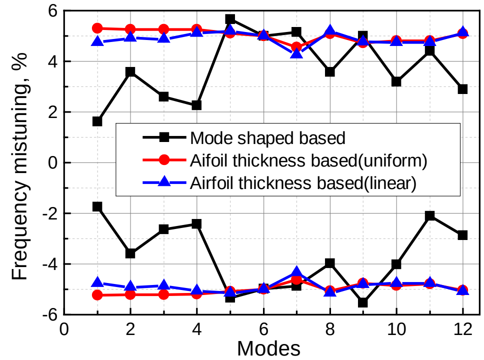


Fig. 7 First 12 natural frequencies mistuning for different element distribution when the 6th mode have ±5% frequency mistuning

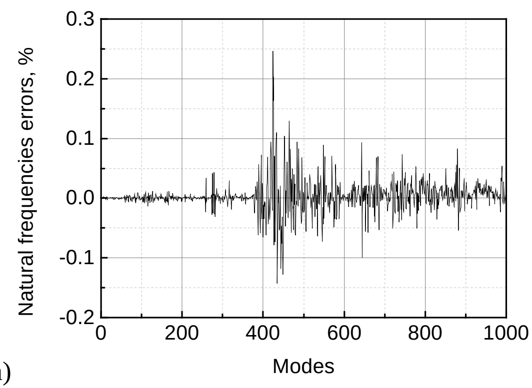
METHOD VALIDATION

The modal properties of a mistuned bladed disc calculated using the proposed method were compared with the results obtained in ANSYS using a full finite element model.

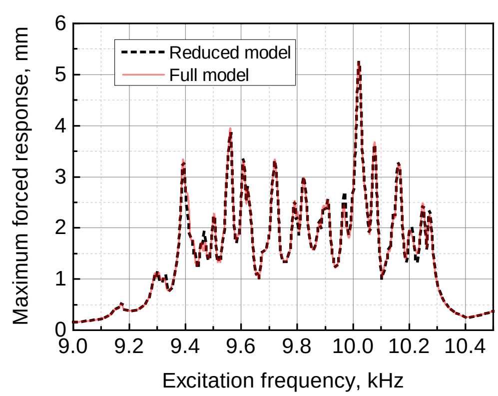
The results of comparison of first 1000 natural frequencies of the mistuned bladed disc are shown in Fig. 8a, where relative errors in natural frequency determination are plotted. It can be seen that the maximum error is less than 0.25% for the first 1000 modes, most of them are less than 0.05%, and it is even less than 0.01% for the first 200 modes. Moreover, the forced response calculated was also compared with results obtained with full model from ANSYS. An example of such a comparison is shown in Fig. 8b, where the maximum forced response selected over all blades and all nodes of the whole bladed disc is plotted. A good correspondence between both methods is achieved as in determination of modal properties and as in the forced response analysis.

NUMERICAL STUDIES AND DISCUSSION

The analysis is performed in most cases using for the basis of model condensation first 32 modes for each nodal diameter, which gives the total number of modes used in the reduced model 32×48=1536, i.e. keeping all modes that were calculated for the tuned bladed disc. The maximum response levels are searched over all blades and over all nodes for each excitation frequency.



a)



b)

Fig. 8 Results the validation: a) errors in natural frequency determination; b) the maximum amplitudes

Forced responses in high-mode frequency ranges

An example of the maximum normalised mistuned forced response performed for a case of 15EO excitation is shown in Fig. 9a. The mistuned forced response is compared here with the forced response of a tuned bladed disc. One can see that the mistuning increases the maximum forced response level by 40% for the considered here a case when the disc is stiff and blade coupling is low. Moreover, mistuning leads to appearance of multitude resonance peaks over a wide frequency range. For the considered case of vibration in the higher modes frequency ranges, the frequency range where resonance peaks occur spans over 1.2 kHz. The maximum normalized blade amplitudes determined for each blade of the bladed disc over the frequency range [9, 10.5 kHz] are shown in Fig. 9b.

It can be seen that there are only 2 blades which have higher response levels than for a tuned system (where all blades have the same response levels). It should be also noted that for the case of 15EO excitation, only the 6th mode is located within the frequency range analysed.

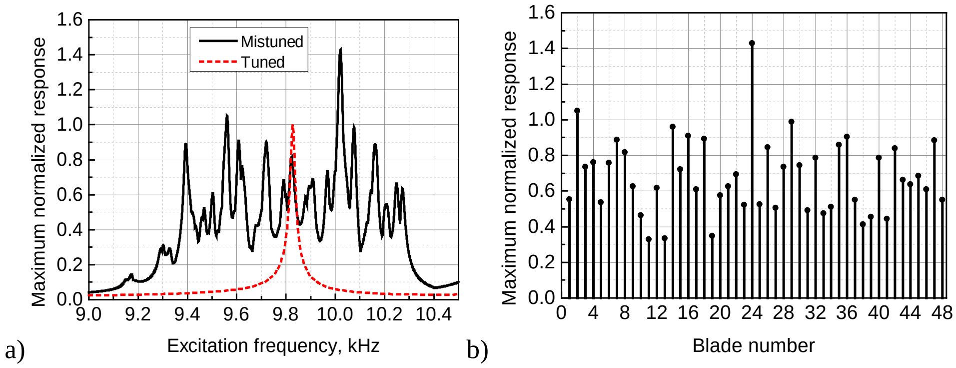


Fig. 9 Normalised forced response excited by 15EO: a) bladed disc maximum response; b) maximum blade amplitudes

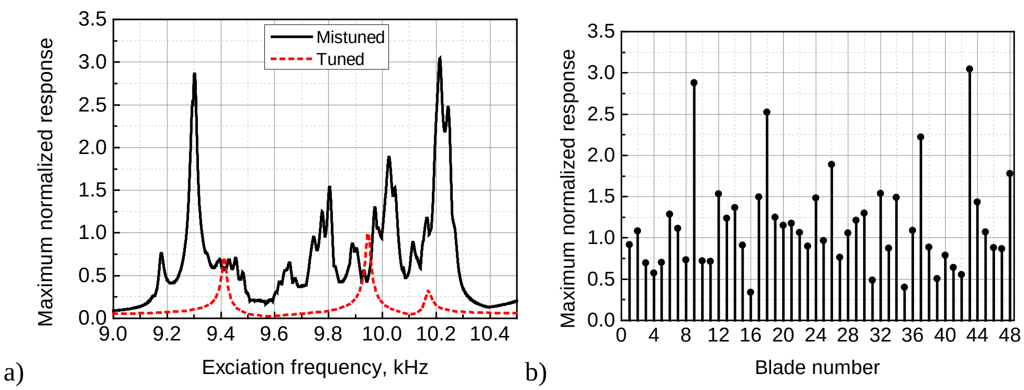


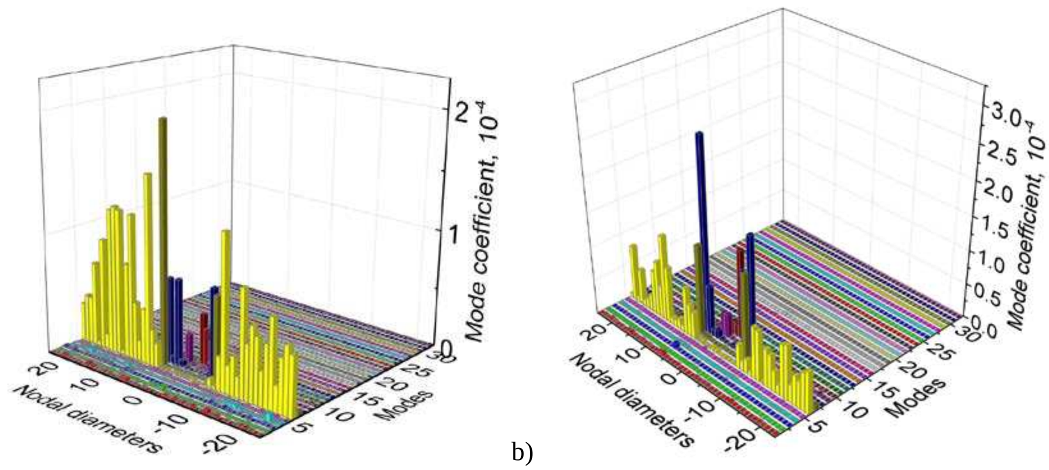
Fig. 10 Normalised forced response excited by 6EO: a) bladed disc maximum response; b) maximum blade amplitudes

For a case of 6EO, natural frequencies of 6th, 7th and 8th modes are close and included in the frequency range examined. The maximum bladed disc forced response is plotted in Fig. 10a.

It can be seen that the mistuning increases the maximum bladed disc forced response amplitudes by factor of 3, if the amplification factor is determined over the whole frequency range analysed here. However, considering the amplitude factor for each of the 3 resonance peaks of the tuned bladed disc, one can see that the amplification factor can even be close to 10 (see resonance peak at 10.16 kHz). The maximum normalized responses for each blade are shown in Fig. 10b, where only a few blades experiencing high amplitude levels can be observed.

To decrease effects of mistuning at the design stage, when a tuned bladed disc design is developed, it is useful to have contributions of different tuned mode shapes in the forced response of the mistuned bladed disc (as derived in Eq.(18)). In Fig. 11, the contributions of each tuned mode in the mistuned forced response excited by 15EO (Fig. 11a) and 6EO (Fig. 11b) are shown.

The contributions are given here for each of 32 modes included in the analysis for each wave number and for each wave number from -24 to +24 and they are calculated at the resonance frequency corresponding to the maximum forced response. For both cases considered here, significant contributions are observed for modes 6, 7, 8, and 9, i.e. for the tuned modes located inside the frequency range considered, and all modes outside provide significantly smaller contributions. It is interesting to note that in both cases the contributions of modes with wave numbers from -5 to +5 are small and contributions of 7th mode families are largest. Moreover, the modes higher than 16 contribute little to the calculated forced response and, therefore, they could be dropped without significant loss of the calculation accuracy. This result corroborates a general heuristic rule that the natural frequencies of modes included in the modal expansion basis should roughly exceed the frequency range analysed by factor 2.



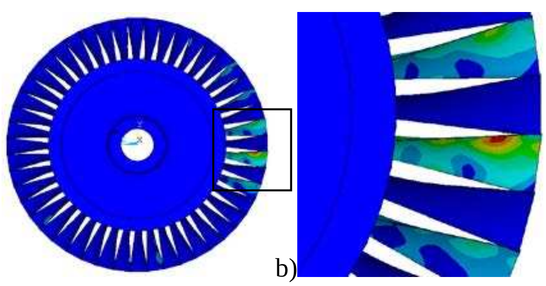
a) b)

Fig. 11 Tuned modes contributions to maximum forced response of the mistuned system: a) 15 EO; b) 6EO

Stress properties in high-mode frequency ranges

In the mistuning studies available in literature the focus is usually set on the determination of displacements amplitudes. Yet, the ultimate characteristics required for assessment of durability and fatigue are stresses occurring in bladed discs. The stress amplitudes are usually proportional to the displacement levels, when we have a resonance vibration regime and the excited mode is well-separated from other resonances. However, these conditions are not satisfied for a mistuned bladed disc where there are often very many different modes interacting at each frequency and the stress analysis need to be performed in addition to the displacement amplitude calculations.

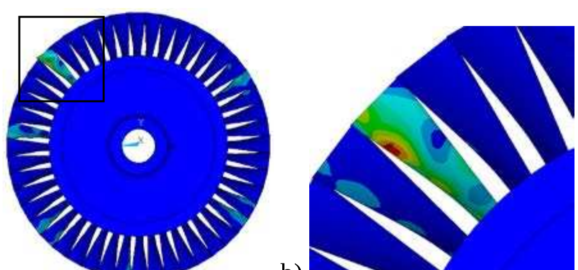
An efficient method for stress calculation for mistuning bladed disc presented in this paper allows calculation of stress distribution using only a sector model. Examples of stress distributions calculated using this method are given in Fig. 12 and Fig. 13 where Von Mises stresses are plotted at the resonance frequency corresponding to maximum response level.



a) b)

Fig. 12 Von Mises stress distribution for 15EO: a) a whole bladed disc; b) a zoomed view of blades with highest stresses

In Fig. 14, the maximum amplitude and the maximum stress for each blade in the selected excitation frequency are compared. In order to compare displacement and stress amplitudes, the normalized stress values are defined similar to the normalized response: they are equal to the maximum mistuned value divided by the maximum tuned value. It can be observed that the stress and response amplification factors are very close and the displacement amplification factor can be used as a good approximation for the stress amplification factor even for case of high mode vibrations, although this conclusion need to be used with caution, since it is restricted to the case of the considered bladed disc design and excitation regimes.



a) b)

Fig. 13 Von Mises stress distribution for 6EO: a) a whole bladed disc; b) a zoomed view of blades with highest stresses

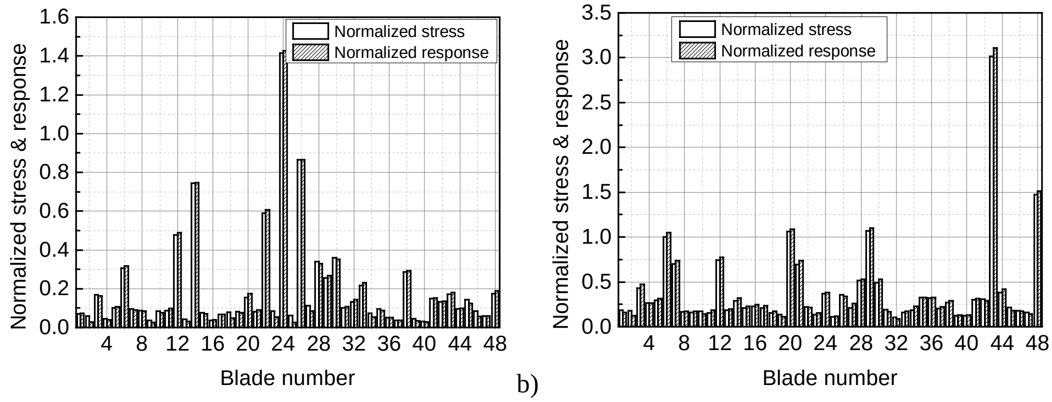


Fig. 14 Stress and displacement amplification factors for each blade at maximum resonance: a) 15EO; b) 6EO

The effects of mistuning elements distribution

Effects of using different distributions of masses simulating blade mistuning on forced response were also explored. In Fig. 15 and Fig. 16, the maximum normalized responses for different distributions of mistuning elements are given. The frequency mistuning pattern was randomly generated in the range [-5%, +5%]. The pattern was controlled by the frequency of 6th blade mode and it was kept the same for all mistuning element distributions. The frequency mistuning pattern.

The plots of maximum responses for each blade are shown in Fig. 17 and Fig. 18. The comparison of results shows that thickness-based mass elements distributed over all blade surface nodes uniformly and linearly distributed provide very close results. The mode-shape-based mass element distribution produces results slightly different from the thickness-based distribution. Yet, this difference is not very significant and it is possible to conclude that, even for vibrations in higher modes shapes, the particular type of mistuning element distribution may not be important and blade natural frequencies can be a sufficient characteristic in order to describe mistuning in bladed discs.

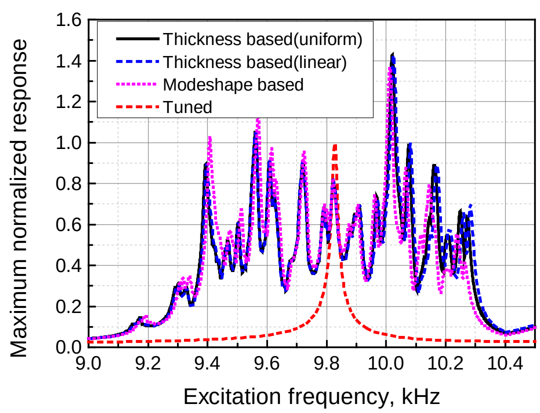


Fig. 15 Maximum normalized response for mistuned and tuned bladed discs excited by 15EO

s

Fig. 16 Maximum normalized response for mistuned and tuned bladed discs excited by 6EO

Mistuned effects for other high mode vibrations

In addition to the reported above analysis, the mistuning effects in higher frequency ranges were examined. The frequency ranges [10.5, 12 kHz] and [12, 13 kHz] were considered, which correspond to a frequency of 7th and 8th mode of a lone blade respectively. Examples of tuned and mistuned forced response for range [10.5, 12 kHz] are given in Fig. 19 for cases of 2EO and 7 EO excitation.

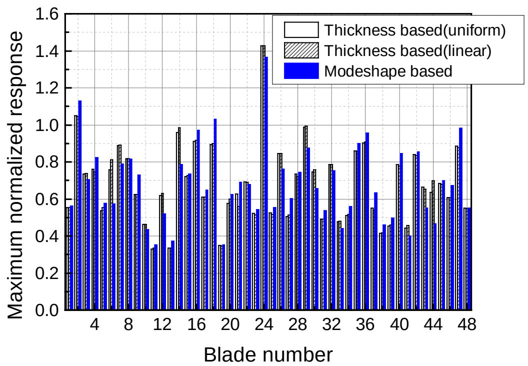


Fig. 17 Maximum normalized response for each blade in the frequency range excited by 15EO

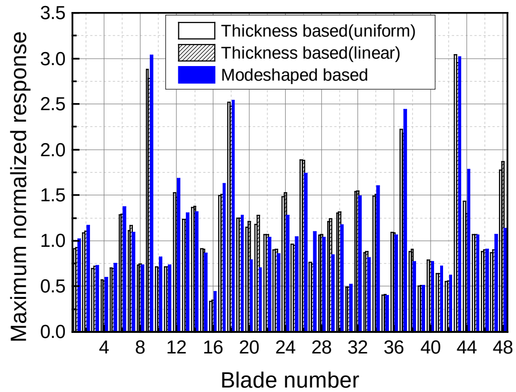
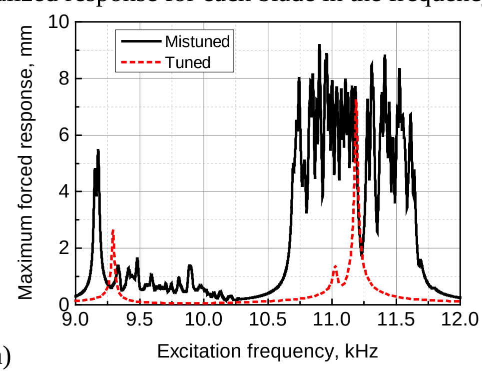
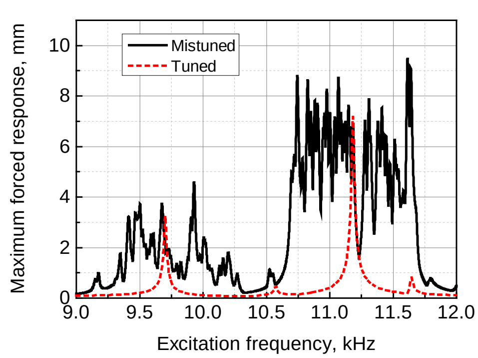


Fig. 18 Maximum normalized response for each blade in the frequency range excited by 6EO



a)



b)

Fig. 19 Maximum forced response for mistuned and tuned bladed discs: a) 2EO; b) 7EO

The plots include also forced response for the first frequency range [9, 10.5 kHz] analysed before. The mistuned analysis was performed for all possible engines order values and the amplification factors obtained for first and second frequency ranges are summarised in Fig. 20. One can see that, for both high mode frequency ranges, the maximum amplification factor values are attained for EOs corresponding to frequency veering regions in Fig. 4.

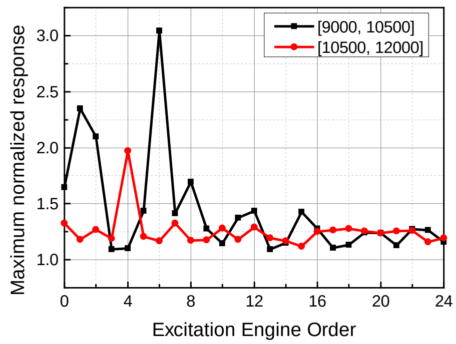


Fig. 20 Maximum normalized response in different high frequency ranges under different EO excitation

Examples of tuned and mistuned forced response for frequency range [12, 13 kHz] are given in Fig. 21 for cases of 5EO and 11 EO excitation. One can see that mistuned bladed disc amplitudes are much higher than for a tuned bladed disc over the whole frequency range. The amplification factors determined for the whole frequency range are 1.7 and 1.8 accordingly.

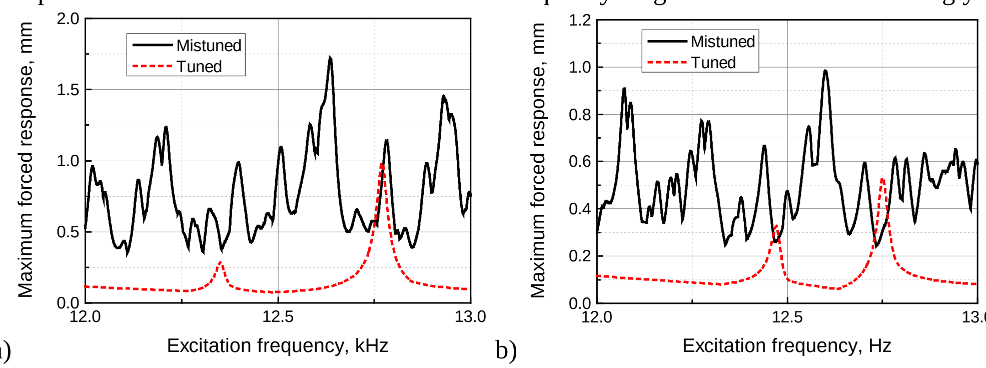


Fig. 21 Maximum forced response for mistuned and tuned bladed discs in other high modes: a) 5EO; b) 11 EO

However, the resonance peak amplitudes of a tuned bladed disc determined for first resonance peaks in Fig. 21 (i.e. at 12.32 kHz for 5EO and 14.6 kHz for 11 EO) are lower than even out-of-resonance responses of a mistuned bladed disc. The amplification factor calculated for these resonance peaks takes very high values, for example it takes value 7.1 for 5EO (see Fig. 21a). So, the analysis based on a tuned forced response cannot provide a sufficiently good estimate for the stress levels in the bladed disc. Moreover, even the Whitehead worst estimate for the amplification factor (Ref. [6]) provides value: $0.5\sqrt{1+N_B} = 3.5$ here, which is much lower than value 7.1 obtained here by the accurate mistuning analysis for high mode vibrations.

Statistical properties of the mistuned response

The Monte Carlo simulations have been performed for the mistuned bladed disc under excitation by 6EO and 15 EO in frequency range [9, 10.5 kHz]. 1000 random bladed disc frequency mistuning patterns were generated where blade mistuning is expressed in terms of 6th natural frequency of alone blade. The random blade mistuning generator provided statistically uniform distribution with ±5% range of maximum deviation of blade natural frequency from its nominal value.

The data obtained as a result of this numerical experiment was sufficient to obtain major statistical characteristics of amplification factors. Here, 6 characteristics were selected: minimum and maximum values, variance (a measure of scatter), mean, skewness (a measure of distribution asymmetry) and kurtosis (a measure of peakedness). These characteristics are given in Table 1.

Table 1 Statistical characteristics of the maximum normalized response of the bladed disc

EO	Total	Mean	Var.	Min.	Max.	Skew.	Kurt.
6	1000	2.633	0.080	1.712	3.770	0.361	3.139
15	1000	1.224	0.013	0.959	1.717	0.702	3.884

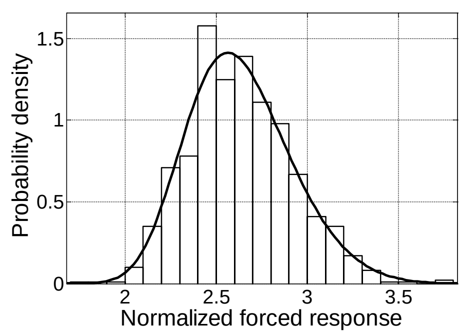
In order to choose the best function which approximates the statistical distributions of the amplification factors, the Kolmogorov-Smirnov test was applied. Two statistical distribution functions were chosen to be tested: (i) the generalized extreme value distribution and (ii) gamma distribution. It is noted here that the distribution functions were tested under 5% significance level condition, which means that there is a 95% chance that the selected theoretical distribution is a genuine probability distribution of the response data. In general, *p* value (see Refs.[23], [24]) is considered as the probability of obtaining a test statistic result at least as close to the one that was actually observed. If the *p*-value is too small (usually less than the significance level) then it suggests that the observed data is inconsistent with the assumption. Table 2 shows that the general extreme value distribution is suitable for both 6EO and 15 EO data, while gamma distribution can fit data only for 6EO.

The histograms for the probability density function of the amplification factors together with the best fit approximation are shown in Fig. 22.

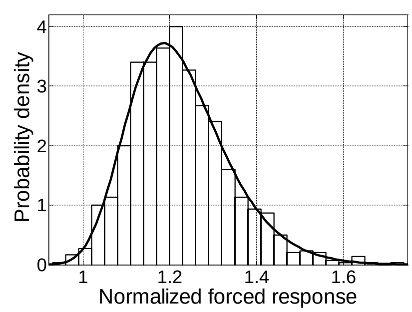
The best-fit general extreme value function for the cumulative probability distribution (CPD) are plotted in Fig. 23 for 6, 14 and 15 EOs. One can see that the 90th percentile value of the normalized response is about 3.0 for 6 EO, and about 1.4 for 15EO. One can see also that the CPDs corresponding to 14 and 15 EOs are almost identical and therefore we can conclude that the statistical behaviour of the amplification factors for these EOs are very close. This can be compared with the results shown in Fig. 20 where all calculation were performed for the same mistuning pattern and where there is significant difference in the amplification factors for 14 and 15 EOs

Table 2 Statistical hypothesis test results for normalized forced response characteristics under different EO excitations

EO	Goodness-of-fit analysis			
	Distribution	Significance level	p-value	Conclusion
6	Generalized extreme value	0.05	0.84	Accept
	Gamma	0.05	0.20	Accept
15	Generalized extreme value	0.05	0.92	Accept
	Gamma	0.05	0.03	Reject



a)



b)

Fig. 22 Statistical distribution of amplification factors for different EOs: a) 6 EO; b) 15EO

CONCLUSIONS

The effects of mistuning have been studied for excitation frequency ranges, corresponding to higher families of bladed disc modes from 6 to 8 and higher. The analysis of such modes represents significant practical interest for the gas-turbine engine industry but has not been performed before.

In order to perform the numerical studies a model reduction method has been developed for computationally efficient analysis of forced response aimed specifically at calculation of high-frequency vibration of mistuned bladed discs.

The method uses high-fidelity models and include new important capabilities: (i) the calculation of high-frequency vibrations; (ii) the calculation of stresses for a mistuned bladed disc and expansion of the results of mistuning analysis for the full finite element model of a bladed disc and (iii) calculation of contributions of tuned mode shapes in the response of a mistuned bladed disc.

New approaches allowing significant reduction of the computation effort necessary of the maximum response calculation have been proposed.

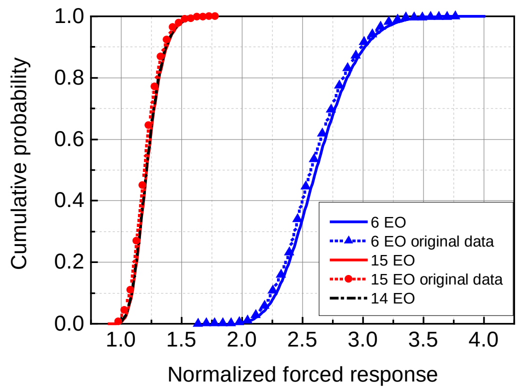


Fig. 23 Best-fit cumulative distribution function for normalized response under different EO excitations

The validation of the method developed has been performed by the comparison of results for modal and forced response analyses with those obtained using commercial finite element software (ANSYS).

The existence of high amplification levels for the high-modes mistuning vibrations has been discovered including cases when such amplitudes are higher than the Whitehead limit.

Moreover, it has discovered that, in some cases, the resonance peak response of a tuned structure can be lower than out-of-resonance amplitudes of a mistuned bladed disc.

The correlation between maximum stresses and displacements has been established for vibration involving a multitude of complex mode shapes. The results show that the stress amplification factor is nearly equal to the amplitude amplification factor. Effects of different modelling of blade frequency mistuning have been examined.

In the end, statistical properties of the forced response of the high-mode vibrations of mistuned bladed discs have been explored using Monte Carlo simulation for which a representative set of 1000 different randomly generated mistuning patterns were considered.

ACKNOWLEDGEMENTS
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